Komplet równań konstytutywnych dla ośrodka liniowo - sprężystego w zagadnieniu trójwymiarowym

3 równania równowagi

$$\sigma_{ij,j} + b_i = \rho \frac{\mathrm{d}^2 u_i}{\mathrm{d}t^2}$$

$$\begin{cases} \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{xy}}{\partial y} + \frac{\partial \sigma_{xz}}{\partial z} + b_x &= \rho \frac{\mathrm{d}^2 u}{\mathrm{d}t^2} \\ \frac{\partial \sigma_{yx}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} + \frac{\partial \sigma_{yz}}{\partial z} + b_y &= \rho \frac{\mathrm{d}^2 v}{\mathrm{d}t^2} \\ \frac{\partial \sigma_{zx}}{\partial x} + \frac{\partial \sigma_{zy}}{\partial y} + \frac{\partial \sigma_{zz}}{\partial z} + b_z &= \rho \frac{\mathrm{d}^2 w}{\mathrm{d}t^2} \end{cases}$$

6 równań geometrycznych

$$\varepsilon_{ij} = \frac{1}{2}(u_{i,j} + u_{j,i})$$

$$\begin{cases} \varepsilon_{xy} &= \frac{1}{2}(\frac{\partial u}{\partial y} + \frac{\partial v}{\partial x}) \\ \varepsilon_{xz} &= \frac{1}{2}(\frac{\partial u}{\partial z} + \frac{\partial w}{\partial x}) \\ \varepsilon_{zy} &= \frac{1}{2}(\frac{\partial w}{\partial y} + \frac{\partial v}{\partial z}) \\ \varepsilon_{xx} &= \frac{\partial u}{\partial x} \\ \varepsilon_{yy} &= \frac{\partial v}{\partial y} \\ \varepsilon_{zz} &= \frac{\partial w}{\partial z} \end{cases}$$

6 związków fizycznych

$$\sigma_{ij} = 2\mu\varepsilon_{ij} + \lambda\sigma_{ij}\varepsilon,$$

gdzie $\varepsilon = \varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}$ oznacza dylatację, a λ, μ stałe Lame zdefiniowane jako

$$\mu = G = \frac{E}{2(1+nu)}, \quad \lambda = \frac{\nu E}{(1+\nu)(1-2\nu)},$$

przy czym ν to liczba Poisson'a, a E - moduł Young'a

$$\begin{cases} \sigma_{xx} &= 2\mu\varepsilon_{xx} + \lambda\varepsilon \\ \sigma_{yy} &= 2\mu\varepsilon_{yy} + \lambda\varepsilon \\ \sigma_{zz} &= 2\mu\varepsilon_{zz} + \lambda\varepsilon \\ \sigma_{xy} &= 2\mu\varepsilon_{xy} \\ \sigma_{xz} &= 2\mu\varepsilon_{xz} \\ \sigma_{zy} &= 2\mu\varepsilon_{zy} \end{cases}$$

lub w postaci zależności odwrotnych

$$\varepsilon_{ij} = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\delta_{ij}\sigma_{kk}$$

$$
\begin{cases}
\varepsilon_{xx} &= \frac{1}{E}(\sigma_{xx} - \nu(\sigma_{yy} + \sigma_{zz})) \\
\varepsilon_{yy} &= \frac{1}{E}(\sigma_{yy} - \nu(\sigma_{xx} + \sigma_{zz})) \\
\varepsilon_{zz} &= \frac{1}{E}(\sigma_{zz} - \nu(\sigma_{yy} + \sigma_{xx})) \\
\varepsilon_{xy} &= \frac{1}{2G}\sigma_{xy} \\
\varepsilon_{xz} &= \frac{1}{2G}\sigma_{xz} \\
\varepsilon_{zy} &= \frac{1}{2G}\sigma_{zy}
\end{cases}
$$